Rotational motion

Kinematics

Radians

The circumference of a circle is π*diameter, but another way to write this is 2π*radius.

An arc ⅔ of the way around the circle would be ⅔ * 2πr long.

We say that this arc is ⅔ * 2π radians.

An arc that is x radians has an arc length of x*r.

So instead of just saying y degrees out of a total 360 degrees in a circle, we can also say x radians out of a total 2π.

Angular distance

We measure the amount of degrees we've turned in terms of radians.

Turning ⅔ of the circle is turning 240 degrees, but it's also $4\pi/3$ radians.

Why radians? Well if you go ⅔ of the way around an arc of a circle, what's the distance you've walked?

That would be $4\pi/3 * r$ where r is the radius of the circle. If we measure this angular distance to be $4\pi/3$, we can easily find the non-angular distance by multiplying radius.

Units

Angular distance is represented as θ, or the greek letter theta.

We actually call the distance we walk around the circle linear distance -- even though it's not a line -- because it's not measured as an angle, but in meters, a unit of length that we can measure with a ruler.

Since θ (angular distance) * r (radius) = x (linear distance), and x and r both have units of meters, θ must be unitless.

However, we measured θ in radians, so which is it?

Radians as a unit

If you said you moved an angle of x around a circle, would you know how much you moved around the circle?

Is it x degrees?

Is it x radians?

We wouldn't know, though the default convention if you don't include a unit is radians since ° is used to represent degree.

To reduce confusion, we can use the units rad. By definition rad = m/m, which reduces to no units at all since they cancel.

Angular velocity

We can also ask the question of how fast we are changing our angular distance.

The amount of radians we move per second is our angular velocity.

Since this is asking the amount of radians we move per second, the units are rad/s, or sometimes represented as just 1/s.

It's represented with ω, a lowercase omega, and $\omega r = v$ just like how $\theta r = x$.

Angular acceleration

Change in angular velocity over time is angular acceleration.

It's represented as α, has units rad/s^2, or $1/s^2$, and $\alpha r = a$.

Direction of angular quantities

We use the right-hand rule where you do a thumbs up, curl your hand in the direction of rotation, and represent the angular quantity in the direction that your thumb points.

The angular quantity is perpendicular to the plane of rotation, and is also perpendicular to all linear vectors of motion at that instant.

Angular kinematics

These kinematic equations are the same as the linear kinematic equations, except angular quantities are involved.

Time, however, does not have an angular component and is preserved between the linear and angular equations.

Rotational second law

Distribution of mass

What's harder to spin? A tennis ball by itself or a 10 meter long rope with a tennis ball attached to the end?

The further away mass is distributed, the harder it is to spin.

Inertia was how resistant something is to moving linearly, so rotational inertia -- or moment of inertia, represented as "I", -- is how resistant something is to spinning.

Thus, masses further away have more moment of inertia.

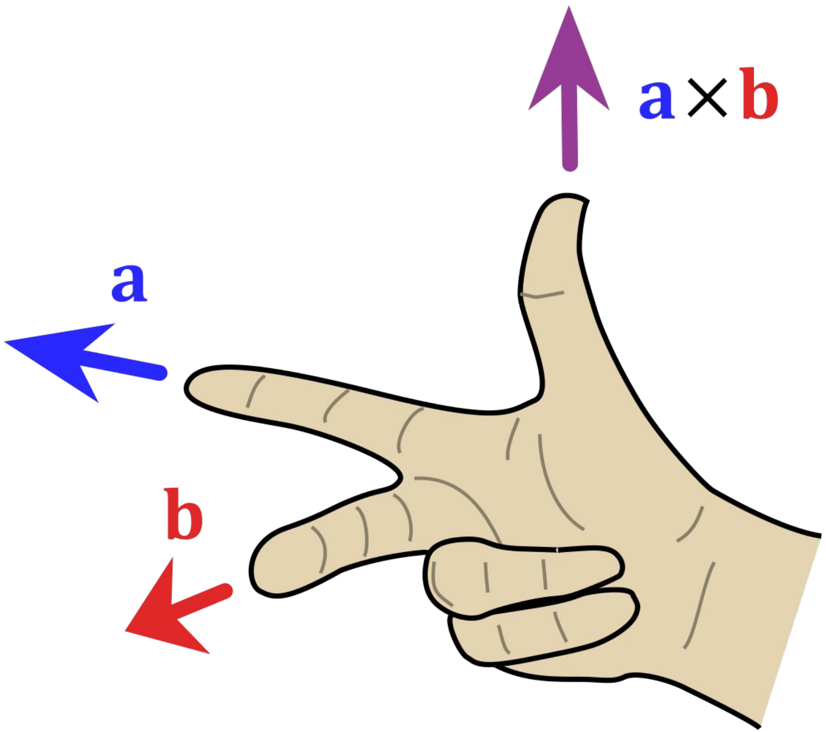
Cross product

There are two ways to multiply vectors: cross products are one of them that return a vector result.

It's represented as **a** x **b** for vectors **a**, **b**, and can only be done for 2, 3, and 7 dimensional vectors with the same dimensions.

The cross product has magnitude |**a**||**b**|sin(θ) where θ is the angle between the vectors.

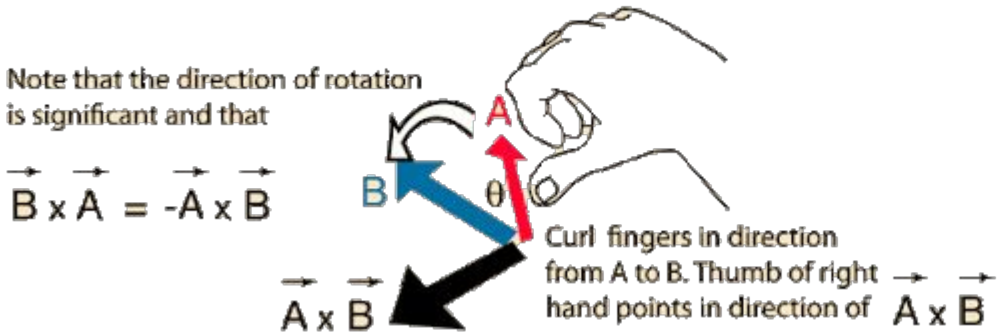
The direction of the vector follows the right hand rule shown on right; however, note that **a** x **b** = -(**b** x **a**).



Torque

Torque, symbol "T," represents how forces translate into angular motion and is a cross product between **r** and **F** where **r** is the direction from the pivot to the lever arm and **F** is the force applied to that pivot.

Note that **r** x **F** has magnitude |**r**||**F**|sin(θ), which really translates to multiplying the component perpendicular to **r** of **F** by the magnitude of **r**.



Rotational Newton’s second law

We know $F = ma$ as Newton’s second law, but we can change this to capture the idea of rotational kinematics.

We can substitute angular quantities for force, mass, and acceleration for the equation $T = I\alpha$.

In other words, torque is moment of inertia times angular acceleration.

Moment of inertia for a point mass

We know for a point a distance r away from the pivot, the torque applied by a force F would be $T = rF$, and as the path this point would make would be circular around the pivot, the angular acceleration would be $\alpha = a/r$.

As $F = ma$, $T = rma$, and as $T = I\alpha$, $I = T/\alpha$, which gives $I = rma/(a/r) = r^2ma/a = r^2m$.

Thus, for a point of mass m a distance r away, $I = mr^2$, which also means I has units of kg * m^2 as mass is measured in kg and the distance r is measured in m.

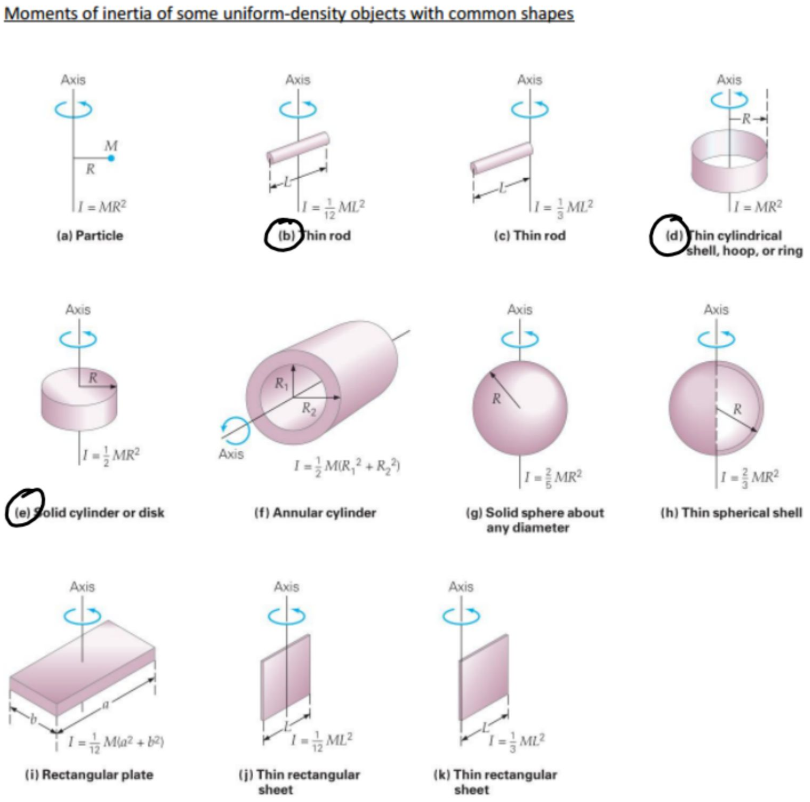
Derivations

All moments of inertia

There are a few objects it's good to know moments of inertia for, including the ones on the right.

However, you should also know how the moments of inertia were derived for the three circled diagrams from the I = mr^2 equation for each point mass.

Image is not from Giancoli, Physics: Principles with Application, but it's a pretty good introductory physics textbook to read which has a moment of inertia table.



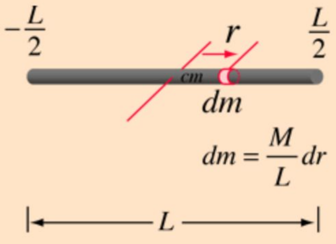
Derivation for a rod

We want to add ρr^2 where ρ is the mass of each point.

If the total mass of the rod of length L is M, then $\rho = M/L$ as we distribute the mass over the entire length of the rod to find the mass of a point.

Then, we integrate $\int \rho r^2 dr$ on both sides of the center of mass (thus from L/2 to the left, or -L/2, and L/2 to the right, or +L/2, which gives us $I_{cm} = ML^2/12$.

Rod Moment Calculation



The moment of inertia calculation for a uniform rod involves expressing any mass element in terms of a distance element dr along the rod. To perform the integral, it is necessary to express eveything in the integral in terms of one variable, in this case the length variable r. Since the total length L has mass M, then M/L is the proportion of mass to length and the mass element can be expressed as shown. Integrating from -L/2 to +L/2 from the center includes the entire rod. The integral is of polynomial type:

Total mass of rod: M

Linear density of rod: $\frac{M}{L}$

Mass of infinitesmal length dr :

$$dm = \frac{M}{L} dr$$

$$I = \int_{-L/2}^{L/2} r^2 \frac{M}{L} dr = \frac{M}{L} \frac{r^3}{3} \Big|_{-L/2}^{L/2} = \frac{M}{3L} \left[\frac{L^3}{8} - \frac{-L^3}{8} \right]$$

$$I_{cm} = \frac{1}{12} ML^2$$

Derivation for a cylinder

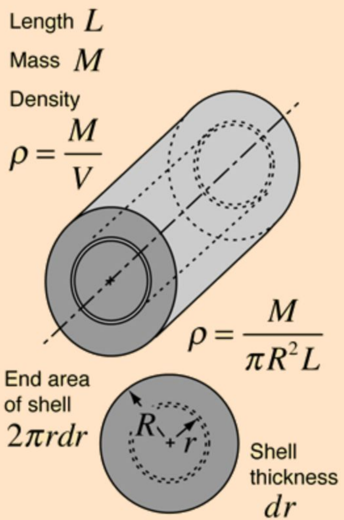
We know ρ for each point is M/V as this is a 3d object, so ρ = M/(base*height of cylinder) or $\rho = M/(\pi R^2/L)$.

The volume of a cylindrical shell is $(\pi a^2 - \pi b^2)h$ for a height L, inner radius b, and outer radius a. This can be rearranged into $\pi L(a^2-b^2)$, or $\pi L(a+b)(a-b)$.

For a shell of inner radius r and outer radius r+dr for infinitesimally small dr, $\pi L[r+(r+dr)][r-(r+dr)] = \pi h(2r+dr)(dr)$ which is close enough to $\pi h(2r)(dr)$ as dr is infinitesimally small.

Moment of Inertia: Cylinder

The expression for the moment of inertia of a solid cylinder can be built up from the moment of inertia of thin cylindrical shells.



Using the general definition for moment of inertia:

$$I = \int_0^M r^2 dm$$

The mass element can be expressed in terms of an infinitesimal radial thickness dr by

$$dm = \rho dV = \rho L 2\pi r dr$$

Substituting gives a polynomial form integral:

$$I = 2\pi\rho L \int_0^R r^3 dr = 2\pi\rho L \frac{R^4}{4}$$

$$I = 2\pi \left[\frac{M}{\pi R^2 L}\right] L \frac{R^4}{4} = \frac{1}{2} MR^2$$

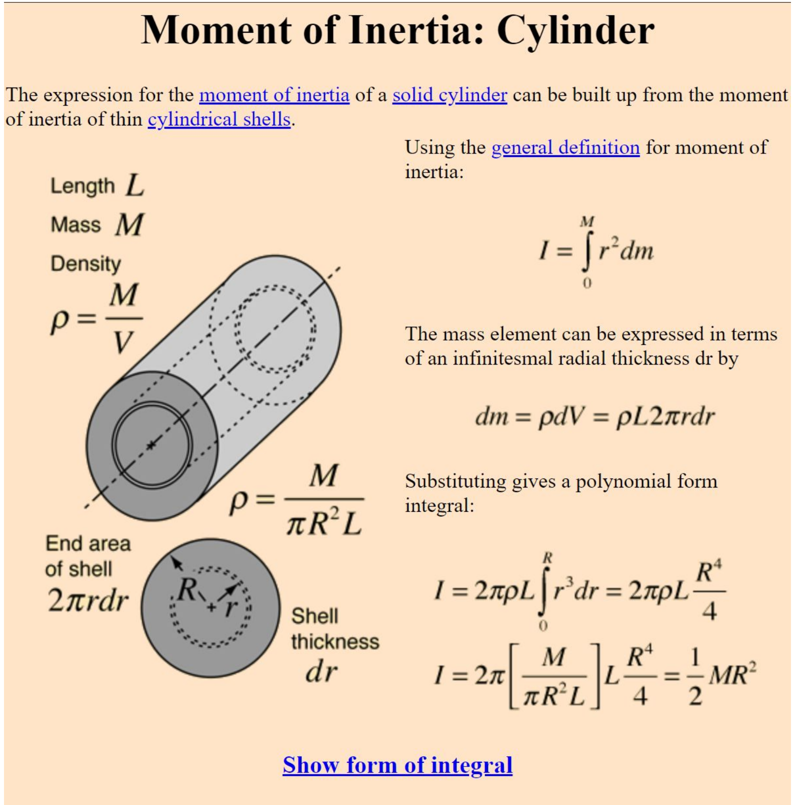
Show form of integral

Derivation for a cylinder cont.

The moment of inertia from each point on the cylindrical shell will be the same, so the total moment of inertia from all masses on that shell is ∫[ρL(2π)r]r^2dr from 0 to R.

Now we substitute in ρ = M/(πR^2L) to get the equation I = ∫[M/(πR^2L)]L(2π)r^3dr, and taking non-r terms out gives I = (2M/R^2)∫r^3dr.

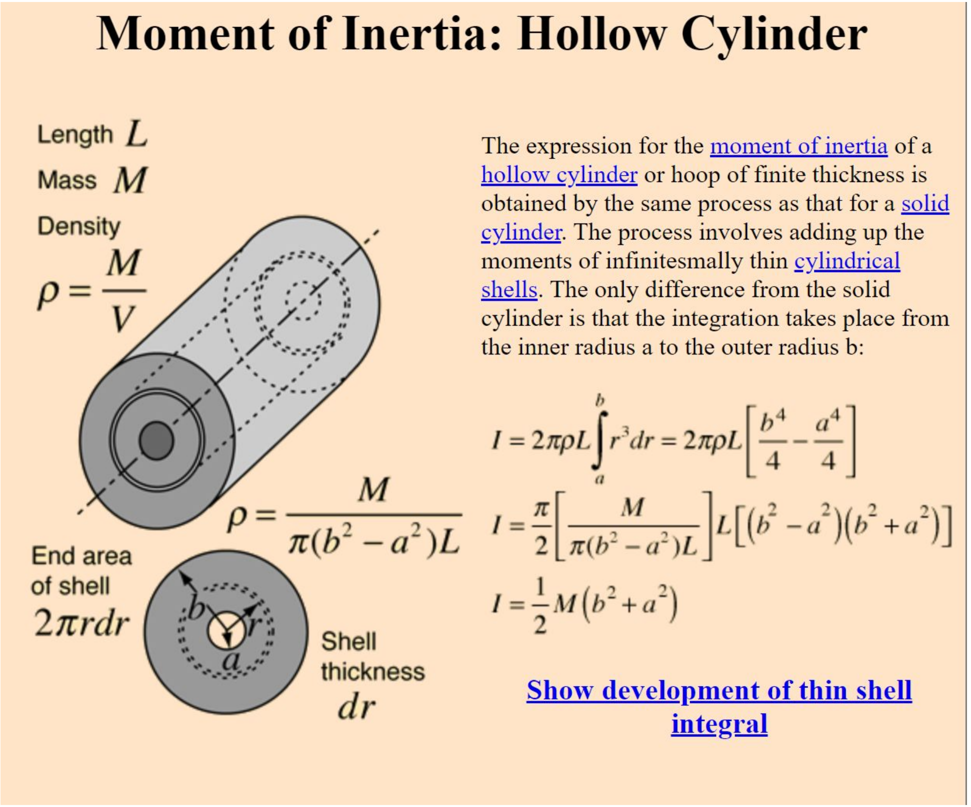
Integrating from 0 to R, aka adding the moments of inertias of each shell, gives the eventual formula I = MR^2/2.



Derivation for a cylindrical shell

Here, we have a cylindrical shell of inner radius a, and outer radius b.

We set up the same integral as the solid cylinder, but instead integrate from a to b instead of from 0 to R to get I = $M(a^2+b^2)/2$.



Moment of Inertia: Hollow Cylinder

The expression for the moment of inertia of a hollow cylinder or hoop of finite thickness is obtained by the same process as that for a solid cylinder. The process involves adding up the moments of infinitesmally thin cylindrical shells. The only difference from the solid cylinder is that the integration takes place from the inner radius a to the outer radius b:

$$I = 2\pi\rho L \int_a^b r^3 dr = 2\pi\rho L \left[\frac{b^4}{4} - \frac{a^4}{4}\right]$$

$$I = \frac{\pi}{2}\left[\frac{M}{\pi(b^2 - a^2)L}\right] L\left[(b^2 - a^2)(b^2 + a^2)\right]$$

$$I = \frac{1}{2} M\left(b^2 + a^2\right)$$

Show development of thin shell integral

Parallel axis theorem

If we know the moment of inertia about an axis touching the center of mass of an object to be I_{cm}, the moment of inertia a distance d away from the center of mass, which we'll call I, is given by $I = I_{cm} + md^2$, where m is the mass of the object.

This allows us to easily calculate rotational inertia from axes that do not intersect the center of mass if we know I_{cm}.

Angular momentum

Equations

Like how $T = r \times F$, angular momentum, represented by L, is given by $L = r \times p$, where p is linear momentum.

However, we also know $p = \int F dt$, so that means $L = \int T dt$.

We also know $p = mv$, so $L = I\omega$.

Symbols and units

Angular momentum is described with a letter L and angular impulse is represented simply as ΔL.

The units for torque are m*N, so the units of angular momentum are N*m*s, or kg*m^2/s.

There are many other ways to derive this as moment of inertia is kg*m^2, and angular velocity is 1/s, so the product of the two gives angular momentum with units kg * m^2/s as well.

Angular momentum with linear motion

If we are watching a race car speed in a straight line from a point not on its path, we will be turning our head to watch it, thus meaning it is rotating and has angular momentum.

If the linear momentum of that race car travelling at a constant velocity is p, then we know L = **r** x **p**.

As **r** x **p** has magnitude |**r**||**p**|sin(θ) where θ is the angle between them, L has that magnitude as well.

Conservation

Angular momentum is also conserved just like linear momentum when no torque is applied.

However, just as if you apply a force, linear momentum can change, the same applies to a system with rotational momentum.

A common example of conservation is a figure skater closing their arms in to reduce their moment of inertia I as mass is closer to the center, thus causing ω to increase to preserve $L = I\omega$, letting them spin faster.